ON THE ALGEBRAIC AND SINGLE-VALUED INTEGRALS IN THE PROBLEM OF MOTION OF A RIGID BODY IN THE NEWTONIAN FORCE FIELD

(OB ALGEBRAICHESKIKH I ODNOZNACHNYKH INTEGRALAKH V ZADACHE O DVIZHENII TVERDOGO TELA V NIUTONOVSKOM POLE SIL)

PMM Vol.27, No.4, 1963, pp.697-698

Iu. A. ARKHANGELSKII
(Moscow)

(Received April 16, 1963)

We shall consider here the problem of motion of a rigid body about a fixed point in the Newtonian central force field, when the fixed point is at the distance R from the center of attraction.

Assuming that the distance R is large as compared with the dimensions of the body, we shall express [1] the force function $U(\gamma, \gamma', \gamma'')$ with accuracy to a constant term in the form of series of powers of R^{-1}

$$\begin{aligned} U(\gamma, \gamma', \gamma'') = &- Mg\,(x_0\gamma + y_0\gamma' + z_0\gamma'') - {}^3/_2 g\,(A\gamma^2 + B\gamma'^2 + C\gamma''^2)\,R^{-1} + \\ &+ {}^3/_2 g\,[(J_{xxx} + J_{xyy} + J_{xzz})\,\gamma + (J_{yyy} + J_{yxx} + J_{yzz})\,\gamma' + (J_{zzz} + J_{zxx} + J_{zyy})\,\gamma'']R^{-2} - \\ &- {}^5/_2 g\,[J_{xxx}\gamma^3 + J_{yyy}\gamma'^3 + J_{zzz}\gamma''^3 + 6J_{xyz}\gamma\gamma'\gamma'' + 3\,(J_{yxx}\gamma'\gamma^2 + J_{zxx}\gamma''\gamma^2 + J_{xyy}\gamma\gamma'^2 + \\ &+ J_{xzz}\gamma\gamma''^2 + J_{zyy}\gamma''\gamma'^2 + J_{yzz}\gamma'\gamma''^2)]\,R^{-2} + (\ldots)\,R^{-3} \end{aligned} \qquad (1)$$

where

$$J_{xyz} = \iiint\limits_V xyz\,\rho dv$$

Let us denote by $U_n(\gamma, \gamma', \gamma'')$ an expression like (1) in which all terms of the order R^{-n} and higher are truncated. From this approximation we obtain the approximate equations of motion of a rigid body about a fixed point in the Newtonian force field

$$A\frac{dp}{dt} + (C - B)\,qr = \gamma''\frac{\partial U_n}{\partial \gamma'} - \gamma'\frac{\partial U_n}{\partial \gamma''}, \qquad \frac{d\gamma}{dt} = r\gamma' - q\gamma'' \qquad \begin{pmatrix} ABC \\ p\,q\,r \\ \gamma\gamma'\gamma'' \end{pmatrix} \qquad (2)$$

which reduce for $n = 0$ to the equations of the conventional problem of motion of a rigid body about a fixed point in a uniform gravitational field.

The right-hand sides of equations (2) are polynomials, and the first integrals of these equations

$$Ap^2 + Bq^2 + Cr^2 - 2U_n(\gamma, \gamma', \gamma'') = C_1 \tag{3}$$
$$Ap\gamma + Bq\gamma' + Cr\gamma'' = C_2, \qquad \gamma^2 + \gamma'^2 + \gamma''^2 = 1$$

are algebraic. Besides, the system of equations (2) does not contain the time t explicitly and has the last Jacobi multiplier equalling unity. Thus equations (2) and their first integrals when $n > 0$, and equations (2) and their first integrals when $n = 0$, have the same properties.

The following [2] well known theorem applies to the equations of motion of a rigid body about a fixed point for the classical problem: the fourth algebraic integral of the equations of motion exists in those cases and in those cases only (in the case of Euler, in the case of Lagrange and in the case of Kovalevska) when the general solutions for p, q, r, γ, γ' and γ'' in the whole complex plane of the variable t are single-valued.

The question arises whether the above theorem applies to equations (2) when $n > 0$.

Let us consider equations (2) when $n = 1$. It has been proved [3,4] that for these equations the fourth algebraic integral exists only in two cases analogous to the cases of Euler and Lagrange for system (2) when $n = 0$. From [5] it follows that other cases at $n = 1$ with single-valued integrals of differential equations (2) are really not new cases, but reduce to the two above mentioned cases.

Solutions for these cases obtained by Kobb [6], Kharlamova [7] and Beletskii [8] failed to explain whether the general integrals of the problem are single-valued or not. There are, however, good reasons to believe that in these two cases there exist single-valued general solutions for p, q, r, γ, γ' and γ'', and the presented theorem is applicable to equations (2) when $n = 1$.

We shall consider now equations (2) when $n = 2$ and we shall prove that the theorem is not applicable in this case. Let us take a rigid body satisfying the conditions

$$x_0 = y_0 = 0, \qquad A = B, \qquad J_{zxx} = J_{zyy}, \qquad J_{zzz} - 3J_{zxx} = l \neq 0$$
$$J_{xxx} = J_{yyy} = J_{xyz} = J_{yxx} = J_{xyy} = J_{yzz} = J_{xzz} = 0$$

(Such a rigid body, for example, would be represented by a homogeneous cylinder of radius r and length h_1. ($h_1 \neq 1/2\sqrt{6}r$) fixed at the center of its base.)

For the latter case the expression for the function U_2 as obtained from formula (1) is

$$U_2(\gamma'') = -Mgz_0\gamma'' - {}^3/_2 g(C-A)\gamma''^2R^{-1} + {}^3/_2 gl\gamma''R^{-2} - {}^5/_2 gl\gamma''^3R^{-2} \qquad (4)$$

and the first three equations of the system (2) become

$$\frac{dp}{dt} - mqr_0 = -\gamma'\varphi, \qquad \frac{dq}{dt} + mpr_0 = \gamma\varphi, \qquad \frac{dr}{dt} = 0$$
$$\left(m = \frac{A-C}{A}, \quad \varphi = \frac{1}{A}\frac{\partial U_2}{\partial \gamma''} = \varphi(\gamma'')\right) \qquad (5)$$

Equations (5) have the fourth integral $r = r_0$, which together with the three first integrals (3) rewritten in the form

$$p^2 + q^2 - \frac{2U_2}{A} = h, \qquad p\gamma + q\gamma' - (m-1)r_0\gamma'' = k, \qquad \gamma^2 + \gamma'^2 + \gamma''^2 = 1 \qquad (6)$$

permit [1] the determination of γ'' from the relation

$$\left(\frac{d\gamma''}{dt}\right)^2 = (1-\gamma''^2)\left[h + \frac{2U_2(\gamma'')}{A}\right] - [k + (m-1)r_0\gamma'']^2 \equiv P(\gamma'') \qquad (7)$$

Separating variables and integrating we obtain

$$t - t_0 = \int \frac{d\gamma''}{\sqrt{P(\gamma'')}} \qquad (8)$$

Inverting the integral (8) gives γ'' as a function of time which would then be the general solution for γ''. From formulas (4) and (7) and from the condition $l \neq 0$ it follows that $P(\gamma'')$ is a fifth degree polynomial of γ''; which means that the integral (8) is a hyperelliptic integral whose inversion [9] is not a single-valued function.

Thus, when $n = 2$ the existence of the fourth general algebraic integral does not imply that this integral is single-valued. By a similar reasoning we can easily find out that this will occur for all $n > 2$ if we consider bodies for which $U_n' = U_n(\gamma'')$.

Consequently, we conclude that the presented theorem is not applicable when $n \geqslant 2$.

BIBLIOGRAPHY

1. Veletskii, V.V., Nekotorye voprosy dvizhenia tverdogo tela v Niutonovskom pole sil (Certain problems of motion of a rigid body in a Newtonian force field). *PMM* Vol. 21, No. 6, 1957.

2. Polubarinova-Kochina, P.Ia., Ob odnoznachnykh resheniakh i algebraicheskikh integralakh zadachi o vrashchenii tiazhelogo tverdogo tela okolo nepodvizhnoi tochki (On the single-valued solutions and algebraic integrals in the problem of rotation of a heavy solid about a fixed point). *Izd. Akad. Nauk SSSR*, 1940.

3. Arkhangelskii, Iu.A., Ob odnoi teoreme Puankare, otnosiashcheisia k zadache o dvizhenii tverdogo tela v Niutonovskom pole sil (On a theorem of Poincare for the problem of motion of a rigid body in the Newtonian force field). *PMM* Vol. 24, No. 6, 1962.

4. Arkhangelskii, Iu.A., Ob algebraicheskikh integralakh v zadache o dvizhenii tverdogo tela v Niutonovskom pole sil (On algebraic integrals in the problem of motion of a rigid body in the Newtonian force field). *PMM* Vol. 27, No. 1, 1963.

5. Arkhangelskii, Iu.A., Ob odnoznachnykh integralakh v zadache o dvizhenii tverdogo tela v Niutonovskom pole sil (On single-valued integrals in the problem of motion of a rigid body in the Newtonian force field). *PMM* Vol. 26, No. 3, 1962.

6. Kobb, G., Sur le probléme de la rotation d'un corps autour d'un point fixe. *Bull. Soc. Math.*, Vol. 23, 1895.

7. Kharlamova, E.I., O dvizhenii tverdogo tela vokrug nepodvizhnoi tochki v tsentralnom Niutonovskom pole sil (On the motion of a rigid body about a fixed point in the central Newtonian force field). *Izv. Sib. Otd. Akad. Nauk SSSR*, No. 6, 1959.

8. Veletskii, V.V., Ob integriruemosti uravnenii dvizhenia tverdogo tela okolo zakreplennoi tochki pod deistviem tsentralnogo Niutonovskogo polia sil (On the integrability of the equations of motion of a rigid body about a fixed point under the action of a central Newtonian force field). *Dokl. Akad. Nauk SSSR*, Vol. 113, No. 2, 1957.

9. Golubev, V.V., *Lektsii po integrirovaniiu uravnenii dvizhenia tiazhelogo tverdogo tela okolo nepodvizhnoi tochki (Lectures on the integration of the equations of motion of a heavy solid about a fixed point)*. Gostekhizdat, 1953.

Translated by T.L.